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STATISTICAL THEORY OF INHOMOGENEOUS TURBULENCE

PART 1

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ABSTRACT. Differential equations for the statistical correlation between two components of velocity variations are derived from the Navier-Stokes equations of motion and the effect of the terms appearing in the equations discussed. Of special significance here are the correlations between pressure variations and the variations of the velocity derivations, the purpose of which is to distribute the variations of velocity uniformly in all directions. A calculated example of a shearing parallel flow demonstrates the interaction of individual effects and makes comparisons with experimental results possible.

1. Introduction

In turbulent flow, in addition to the Navier-Stokes equations for the average flow motion, the base flow, other relationships, defining the connection between the turbulent stresses which occur in the equations of motion, the so-called Reynolds stresses, and the rest of the flow values are needed for calculation of the flow processes. Such a relationship is, e.g., the statistical balance given in a work by L. Prandtl [1] for the total kinetic energy of turbulent motion, as derived from the Navier-Stokes equations. The following discussion continues the work of L. Prandtl, but takes another step in the analysis of turbulent motion by individually considering the statistical equilibrium of the energy contained in the three components of the rate of variations, which act perpendicularly to each other. A new effect, the exchange of energy between the different velocity components, is introduced. In addition, differential equations are derived for the correlations which exist between two different velocity components; the equations demonstrate the causes of the correlations. /547*

Due to the correlations between pressure and velocity variations and the three velocity components, no formal mathematical treatment of the differential

*Numbers in the margin indicate pagination in the foreign text.

equations is possible. It is feasible to derive differential relationships for tertiary correlations, etc. by repeated multiplication of the Navier-Stokes equations of motion with one of the velocity components. However, further development of this not entirely new idea does not lead to a solvable system of equations, because the number of unknowns arising in the form of correlations of higher degrees is greater than the number of equations obtained. The further processing of the equations obtained thus requires additional physical considerations, which in the present work shall be expressed by semiempirical statements. /548

2. Differential Equations for the Components of the Correlation Tensor

Let us first derive formally the relationships required in the considerations. For this purpose, x_i Cartesian coordinates shall be introduced with $i = 1, 2, 3$. Let U_i be the velocity component of the base flow (average value)[1], u_i the components of the rate of variation. Further, let P be the average value of pressure and p the variation in pressure. Only constant volume flow shall be considered. The continuity condition must be satisfied both by the base flow and the variation motion:

$$\sum_{i=1}^{3} \frac{\partial U_i}{\partial x_i} = 0; \qquad \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0. \tag{2.1}$$

Then, in general, the Navier-Stokes equations of motion are the following:

$$\begin{aligned} \frac{\partial}{\partial t}(U_i + u_i) + \sum_{k=1}^{3}(U_k + u_k)\frac{\partial}{\partial x_k}(U_i + u_i) \\ = -\frac{1}{\varrho}\frac{\partial}{\partial x_i}(P + p) + \nu \Delta (U_i + u_i). \end{aligned} \tag{2.2}$$

Averaging and combination of the stress components of molecular and turbulent friction (Reynolds stresses) in an average stress tensor yields the following expression for the components of this tensor: /549

$$\pi_{ik} = \varrho\left[\nu\left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i}\right) - \overline{u_i u_k}\right]; \tag{2.3}$$

1. In the case of nonstationary spatially inhomogeneous flow, neither the usual time averaging nor space averaging lead to a statistical description of flow characteristics satisfactory with respect to theory. Averaging over a large number of independent systems in which the flow processes under consideration are assumed to occur under uniform conditions, yields more suitable results.

and Eq. (2.2) may be written in the following familiar and clearer form:

$$\frac{\partial U_i}{\partial t} + \sum_{k=1}^{3} U_k \frac{\partial U_i}{\partial x_k} = -\frac{1}{\varrho}\frac{\partial P}{\partial x_i} + \frac{1}{\varrho}\sum_{k=1}^{3}\frac{\partial \pi_{ik}}{\partial x_k} \tag{2.4}$$

Altogether, there are three ($i = 1, 2, 3$) such equations of motion. The excess unknowns are the components of the correlation tensor

$$\mathfrak{R} = \begin{pmatrix} \overline{u_1^2} & \overline{u_1 u_2} & \overline{u_1 u_3} \\ \overline{u_2 u_1} & \overline{u_2^2} & \overline{u_2 u_3} \\ \overline{u_3 u_1} & \overline{u_3 u_2} & \overline{u_3^2} \end{pmatrix} \tag{2.5}$$

these are the tensor components of the Reynolds stresses divided by ρ.

A similar system of equations for the six unknowns $\overline{u_i u_j}$ may be obtained by multiplying Eq (2.2) by u_j and averaging. This formally yields:

$$\begin{aligned} \overline{u_j \frac{\partial u_i}{\partial t}} + \sum_{k=1}^{3} U_k \overline{u_j \frac{\partial u_i}{\partial x_k}} + \sum_{k=1}^{3} \overline{u_k u_j \frac{\partial u_i}{\partial x_k}} + \sum_{k=1}^{3} \overline{u_k u_j}\frac{\partial U_i}{\partial x_k} \\ = -\frac{1}{\varrho}\overline{u_j \frac{\partial p}{\partial x_i}} + \nu \overline{u_j \Delta u_i}. \end{aligned} \tag{2.6}$$

The addition to this relationship of a corresponding equation obtained from Eq. (2.6) by interchanging the indices i and j yields the desired system. In order to determine the physical meaning of the individual terms to create the conditions for semiempirical statements, certain transformations are required. Thus

$$\sum_{k=1}^{3} \overline{\frac{\partial u_k u_i u_j}{\partial x_k}} = \sum_{k=1}^{3} \overline{\frac{\partial u_k}{\partial x_k} u_i u_j} + \sum_{k=1}^{3} \overline{u_k \frac{\partial u_i}{\partial x_k} u_j} + \sum_{k=1}^{3} \overline{u_k u_i \frac{\partial u_j}{\partial x_k}}, \tag{2.7}$$

where the first term of the right side of Eq. (2.7), due to the continuity Eq. (2.1) is equal to 0. Further

$$\Delta \overline{u_i u_j} = 2 \sum_{k=1}^{3} \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} + \overline{u_i \Delta u_j} + \overline{u_j \Delta u_i} \tag{2.8}$$

and

$$\frac{\partial \overline{u_j p}}{\partial x_i} = \overline{p \frac{\partial u_j}{\partial x}} + \overline{u_j \frac{\partial p}{\partial x_i}}. \tag{2.9}$$

Application of these relationships yields /550

$$
\left.\begin{aligned}
&\frac{\partial \overline{u_i u_j}}{\partial t} + \sum_{k=1}^{3} U_k \frac{\partial \overline{u_i u_j}}{\partial x_k} + \sum_{k=1}^{3} \overline{u_k u_j} \frac{\partial U_i}{\partial x_k} + \\
&+ \sum_{k=1}^{3} \overline{u_k u_i} \frac{\partial U_j}{\partial x_k} - \frac{1}{\varrho} \overline{p \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)} + \\
&+ \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \left[-\nu \frac{\partial \overline{u_i u_j}}{\partial x_k} + \overline{u_k u_i u_j} + \overline{(\delta_{jk} u_i + \delta_{ik} u_j) \frac{p}{\varrho}} \right] + \\
&+ 2\nu \sum_{k=1}^{3} \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}} = 0.
\end{aligned}\right\} \qquad (2.10)
$$

Here, δ_{ik} is the Kronecker symbol ($\delta_{ik} = 1$ for $i = k$, $\delta_{ik} = 0$ for $i \neq k$). A total of six various equations of the Eq. (2.10) type can be obtained by selecting all possible combinations of i and j. Physical meanings can be discussed in the most illustrative manner for i = j. Multiplication by $\varrho/2$ yields for this case

$$
\left.\begin{aligned}
&\underbrace{\frac{\varrho}{2} \frac{\partial \overline{u_i^2}}{\partial t} + \frac{\varrho}{2} \sum_{k=1}^{3} U_k \frac{\partial \overline{u_i^2}}{\partial x_k}}_{\text{Total change in kinetic energy}} + \sum_{k=1}^{3} \underbrace{\varrho \overline{u_k u_i} \frac{\partial U_i}{\partial x_k}}_{\text{Work of Reynolds stresses}} \underbrace{- \overline{p \frac{\partial u_i}{\partial x_i}}}_{\text{Energy exchange with other variation components}} + \\
&\underbrace{\sum_{k=1}^{3} \frac{\partial}{\partial x_k} \left[-\frac{\varrho \nu}{2} \frac{\partial \overline{u_i^2}}{\partial x_k} + \overline{u_k \left(\frac{\varrho}{2} u_i^2 + \delta_{ik} p \right)} \right]}_{\text{Diffusion of } \frac{\varrho}{2}\overline{u_i^2}} + \underbrace{\varrho \nu \sum_{k=1}^{3} \overline{\left(\frac{\partial u_i}{\partial x_k} \right)^2}}_{\text{Dissipation}} = 0.
\end{aligned}\right\} \qquad (2.11)
$$

The normal Reynolds stresses of $-\varrho \overline{u_i^2}$ may be interpreted as the double energy content of the u_i variations. Eq. (2.11) thus describes the equilibrium of kinetic energy contained in a single variation component. The first two terms of Eq. (2.11) state the substantial variation in time of the energy component $\varrho/\overline{u_i^2}2$. The third term represents the work component of the Reynolds stresses, which is being converted in the kinetic energy of the variation component u_i. The remaining terms, which will be discussed in more detail later, express the exchange of energy, energy dissipation with other variation components and the exchange of energy between different locations of the slow space (diffusion of energy).

The energy contributions to the variation components are scalar magnitudes. The three Eq. (2.11) for the indices i = 1, 2, 3 can therefore be summed and thus combined into a balance for the total variation energy. With an average energy relative to unit mass /551

$$E = \tfrac{1}{2}\sum_{i=1}^{3}\overline{u_i^2} \tag{2.12}$$

therefore

$$\varrho\left[\frac{\partial E}{\partial t} + \sum_{i=1}^{3} U_i \frac{\partial E}{\partial x_i}\right] + \sum_{i=1}^{3}\sum_{k=1}^{3} \varrho\, \overline{u_i u_k}\, \frac{\partial U_i}{\partial x_k} + \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\left[-\varrho\nu \frac{\partial E}{\partial x_k} + \overline{u_k\left(\varrho\sum_{i=1}^{3}\frac{u_i^2}{2} + p\right)}\right] + \varrho\nu \sum_{i=1}^{3}\sum_{k=1}^{3}\overline{\left(\frac{\partial u_i}{\partial x_k}\right)^2} = 0. \tag{2.13}$$

This equation expresses Prandtl's argument in [1]. The term representing the energy exchange of the velocity components among each other, no longer appears in this equation.

For unequal indices $i \neq j$, Eq. (2.10) presents a differential equation for the correlation between two variation components u_i and u_j, acting perpendicularly upon each other. This $\overline{u_i, u_j}$ correlation is subject to effects similar to those of the kinetic energy. The first two terms of Eq. (2.10) describe variation in time and the convection of $\overline{u_i u_j}$. The sums of the third and fourth term state that existing variation components and potentially existing correlations between these, produce new correlations by way of the convection of fluid particles if the base flow is inhomogeneous. Thus, the square of variation $\overline{u_i^2}$ and the gradient $\partial U_i/\partial x_i$ furnishes a negative contribution to the variation in time of the $\overline{u_i u_j}$ correlation

$$\frac{\partial \overline{u_i u_j}}{\partial t} = -\overline{u_j^2}\,\frac{\partial U_i}{\partial x_j}. \tag{2.14}$$

In the same manner, a negative contribution to $\partial\overline{u_i u_j}/\partial t$ is obtained if a positive statistical correlation exists between the variation component u_j and a third component u_k and if simultaneously a $\partial U_i/\partial x_k$ gradient also exists. Additional study shows that the generation of correlations between two components acting perpendicularly upon each other can always be traced to effects of this type. $\overline{u_i u_j}$ correlations can therefore be formed only if the base flow is nonuniform.

The physical action of the rest of the terms in Eq. (2.10) cannot be explained as clearly as in the energy equations, but is closely related to the action of the corresponding terms in Eq. (2.11).

A few additional equations are needed for the connection between pressure and velocity. Differentiation of Eq. (2.2) with respect to x_i and summation over all i values yields, with consideration of the continuity condition in Eq. (2.1), the following known expression for pressure /552

$$\frac{1}{\varrho}\Delta(P+p) = -\sum_{k=1}^{3}\sum_{i=1}^{3}\frac{\partial(U_k+u_k)}{\partial x_i}\,\frac{\partial(U_i+u_i)}{\partial x_k} \qquad (2.15)$$

/2

Averaging leads to

$$\frac{1}{\varrho}\Delta P = -\sum_{k=1}^{3}\sum_{i=1}^{3}\left(\frac{\partial U_k}{\partial x_i}\,\frac{\partial U_i}{\partial x_k} + \frac{\partial^2 \overline{u_k u_i}}{\partial x_k \partial x_i}\right). \qquad (2.16)$$

Subtraction of this value from Eq. (2.15) results in the following expression for pressure variations:

$$\frac{1}{\varrho}\Delta p = -\sum_{k=1}^{3}\sum_{i=1}^{3} 2\,\frac{\partial U_i}{\partial x_k}\,\frac{\partial u_k}{\partial x_i} - \sum_{k=1}^{3}\sum_{i=1}^{3}\left(\frac{\partial^2 u_k u_i}{\partial x_k \partial x_i} - \frac{\partial^2 \overline{u_k u_i}}{\partial x_k \partial x_i}\right). \qquad (2.17)$$

These variations of pressure satisfy Poisson's differential equation; the expression at the right side of Eq. (2.17) represents the proof. For stationary points located not too closely to the wall with a space vector of $\mathfrak{x}$, Green's law yields the following:

$$\begin{aligned}\frac{1}{\varrho}p(\mathfrak{x}) = {}& \frac{1}{2\pi}\sum_{k=1}^{3}\sum_{i=1}^{3}\int_{(\mathrm{Vol})}\frac{\partial U_k(\mathfrak{x}+\mathfrak{r})}{\partial x_i}\,\frac{\partial u_i(\mathfrak{x}+\mathfrak{r})}{\partial x_k}\,\frac{d\,\mathrm{Vol}}{r} + \\ &+ \frac{1}{4\pi}\sum_{k=1}^{3}\sum_{i=1}^{3}\int_{(\mathrm{Vol})}\left[\frac{\partial^2 u_k u_i}{\partial x_k \partial x_i}(\mathfrak{x}+\mathfrak{r}) - \frac{\partial^2 \overline{u_k u_i}}{\partial x_k \partial x_i}(\mathfrak{x}+\mathfrak{r})\right]\frac{d\,\mathrm{Vol}}{r} \\ &\text{with } r = |\mathfrak{r}| = \sqrt{\sum_{i=1}^{3}\xi_i^2} \quad\text{and}\quad d\,\mathrm{Vol} = d\xi_1\,d\xi_2\,d\xi_3.\end{aligned} \qquad (2.18)$$

The variations of pressure thus are due to a component proportional to the base flow velocities and a component generated by interactions between different variation components. Eq. (2.18) will be useful in further investigation.

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2. Here $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ represents the Laplace operator.

3. Energy Interchange Between Different Velocity Components /553

Let us initially consider the correlations between pressure variations and the variations of the velocity gradient $\overline{p\frac{\partial u_i}{\partial x_i}}$. Due to the continuity condition in Eq. (2.1)

$$\sum_{i=1}^{3}\overline{p\frac{\partial u_i}{\partial x_i}} = \overline{p\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}} = 0. \tag{3.1}$$

These terms therefore do not contribute to the total balance of kinetic energy. In the derivation of the energy equation (2.13) for all three variation components acting perpendicularly to each other, this is expressed by the elimination of the $\overline{p\frac{\partial u_i}{\partial x_i}}$ terms. They effect only an interchange of energy between the different variation components. If the contributions due to the base flow are initially neglected, such an exchange may be imagined approximately in the following manner:

If two elements of turbulence (or turbulence spheres) move towards the 0 point, from different sides, parallel to the x_i axis (Figure 1), the fluid between them will be displaced. If, at the same time, a pressure maximum exists at 0, the u_i component must do work and thus loses some of its kinetic energy. The u_j component, on the other hand, experiences acceleration. In this case, therefore, the u_i component loses energy to the u_j component. Since in the example $\partial u_i/\partial x_i$ is negative at 0, $\overline{p\frac{\partial u_i}{\partial x_i}}$ is also negative. This consideration demonstrates clearly that in the case of a negative $\overline{p\frac{\partial u_i}{\partial x_i}}$, the u_i component transfers energy to the other components. With a positive $\overline{p\frac{\partial u_i}{\partial x_i}}$, on the other hand, the u_i component receives energy from the other components.

It should be recalled in connection with the above considerations that turbulence spheres have no solid boundaries and only a limited life time. They thus do not represent immutable parts as e.g. the molecules of a gas. Specifically, the model of rigid spheres, so successful in the kinetic theory of gases which collide elastically, cannot be used. If two turbulence spheres, as shown in Figure 1, approach each other and, finally, "collide" with each other, they definitely lose their identity. The fluid mass which has been moving

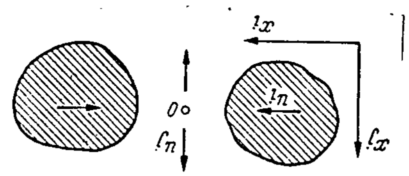

Figure 1. Meeting of two turbulence spheres.

uniformly, now flows in different directions. The kinetic energy, which is being removed from the u_i component through the instantaneous occurrence of positive pressure at location 0, is distributed over two perpendicularly acting components, so that the energy imparted to the individual components is in general less than the energy removed from the component in the direction of the impact. This consideration permits conclusions concerning the sign of the exchange of energy. If, e.g., the square of the average value $\overline{u_i^2}$ is greater than the corresponding values of the other components, the collision in the x_i direction will probably be more violent than in the other directions. The energy transferred from the x_i direction to the velocity variations of other directions will be greater in its statistical average than the energy transmitted to the u_i component by impacts perpendicular to the x_i direction, because only a part of the energy exchanged benefits the x_i direction. It follows from this consideration that the $\overline{p \frac{\partial u_i}{\partial x_i}}$ correlation effects an exchange of energy from the component with higher intensity to the component with lower intensity. The $\overline{p \frac{\partial u_i}{\partial x_i}}$ terms thus impart a tendency to the turbulence to distribute the kinetic energy uniformly over all of the variation components. This effect has already been pointed out by G. K. Batchelor [2] in the special case of axial symmetry turbulence. /554

The simplest initial statement for a formal description of this effect needed for further investigation, is based on the fact that the energy transmitted in unit time from the u_i variation to the u_j variation is proportional to the difference in energy contained in these components: $\frac{\varrho}{2}(\overline{u_i^2} - \overline{u_j^2})$. Since the $\overline{p \frac{\partial u_i}{\partial x_i}}$ expression describes the entire loss of energy suffered by the u_i variations in favor of the other velocity variations in unit time, $\overline{p \frac{\partial u_i}{\partial x_i}}$ is proportional to

$$\varrho\left[\overline{u_i^2} - {\sum_{j=1}^{3}}' \frac{\overline{u_j^2}}{2}\right] = \varrho\left[\frac{3}{2}\overline{u_i^2} - E\right],$$

where the line at the summation sign is intended to indicate that the case of j = i is to be omitted in the summation. In order to obtain a formula for the quantitative determination of the exchange of energy between the different components, the expression obtained must be supplemented by a multiplier which is independent of the i index and which gives the formula desired its correct dimension for $\overline{p \frac{\partial u_i}{\partial x_i}}$, i.e. /555

$$\text{Density} \cdot \frac{(\text{Velocity})^{3/2}}{\text{Length}}$$

This multiplier thus can be only $\sqrt{E}/L$ times a pure number, where L is a length characterizing the average size of turbulence elements. This yields

$$\overline{p\frac{\partial u_i}{\partial x_i}} = -k_p \frac{\varrho}{2} \frac{\sqrt{E}}{L}\left(\overline{u_i^2} - \frac{2}{3}E\right). \tag{3.2}$$

The magnitude of the empirical numerical factor k_p, which depends on the structure of the turbulence, will be indicated by experimental results discussed in Section 6. If the intensities of all three variation components are equal (as in isotropic turbulence), then Eq. (3.2) does not result in an exchange of energy, as would be expected.

Let us now undertake an interpretation and estimate of the $\overline{p\left(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i}\right)}$ terms for i ≠ j. Let us assume that two turbulence elements are approaching point 0 in the manner shown in Figure 2; here again, the fluid in between is being forced out. If a pressure maximum exists at 0, it is known from the foregoing that u_η will gain in energy at the expense of the u_ξ component. The u_ξ component furnishes a positive, u_η a negative contribution to the $\overline{u_i u_j}$ correlation; if, $u_\xi > u_\eta$, then the sum of the contribution of the two components to $\overline{u_i u_j}$ is positive. The transfer of energy from the u_ξ component to u_η reduces the positive contribution of u_ξ to $\overline{u_i u_j}$ and increases the negative contribution of u_η; the result is a negative contribution to $\partial\overline{u_i u_j}/\partial t$.

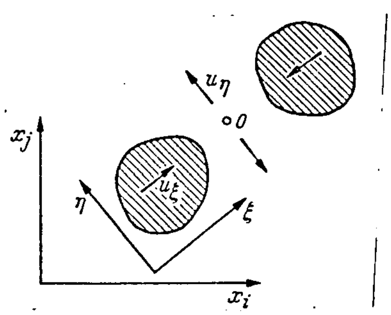

Figure 2. The determination of $p\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$.

In the situation shown in the figure, both $\partial u_i/\partial x_j$ and $\partial u_j/\partial x_i$ are negative and thus $p\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_i}\right)$ is also negative. This explains how a positive average value of $\overline{p\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$ produces a negative contribution to $\partial\overline{u_i u_j}/\partial t$ as expressed in Eq. (2.10).

In addition, with the aid of the fact that the transfer of energy always occurs from a variation component larger on the average to smaller components, it may be seen that in the case of a positive $\overline{u_i u_j}$, a negative $\overline{p\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$

must be expected; as a result, the term under discussion tends to decrease an existing $\overline{u_i u_j}$ correlation. This conclusion supplements statements made concerning the effect of $\overline{p\frac{\partial u_i}{\partial x_j}}$ so that it may be stated in a general manner: correlations between pressure and velocity derivations effect a tendency of turbulence to isotropy. This may also be formulated as follows: in any field of turbulence left to itself, isotropic distribution of velocity variations is the most likely. An anisotropy of turbulence can be produced or maintained by external effects only, e.g., by a superposed inhomogeneous base flow. /556

Quantitative determination of the term $\overline{p(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})}$ begins with Eq. (3.2) Let us consider initially only variations with vectors coinciding with the ξ and η axes (with a small $d\vartheta$ scatter), Figure 3. If $\overline{u_\xi^2}$ and $\overline{u_\eta^2}$ are the squares of the averages of the variations, then their contribution to the correlation $\overline{u_i u_j}$ is:

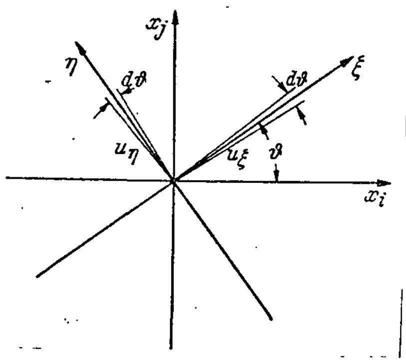

Figure 3. The $\overline{u_i u_j}$ correlation.

$$\delta \overline{u_i u_j} = (\overline{u_\xi^2} - \overline{u_\eta^2}) \frac{\sin 2\vartheta}{2}. \qquad (3.3)$$

Due to the inequality of the variation components u_ξ and u_η the kinetic energy of these components varies; their difference may be calculated by Eq. (3.2) as follows:

$$\frac{\varrho}{2}\frac{\partial}{\partial t}(\overline{u_\xi^2} - \overline{u_\eta^2}) = \overline{p\left(\frac{\partial u_\xi}{\partial \xi} - \frac{\partial u_\eta}{\partial \eta}\right)} = -k_p \frac{\varrho}{2}\frac{\sqrt{E}}{L}(\overline{u_\xi^2} - \overline{u_\eta^2}) \qquad (3.4)$$

Multiplication of these equations by $\frac{\sin 2\vartheta}{2}$, division by $\varrho/2$ and substitution of Eq. (3.3) yields:

$$\sum_{j=1}^{3} a_{jj}^{mi} = 2R_i^m(0) = 2\overline{u_i u_m}. \qquad (3.5)$$

Following integration over the entire angular range, if Eq. (3.2) is considered valid and $i \neq j$:

$$\frac{1}{\varrho}\overline{p\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} = -k_p \frac{\sqrt{E}}{L}\overline{u_i u_j}, \qquad (3.6)$$

where the numerical factor k_p is identical with that of Eq. (3.2). In an entirely general manner, the following expression is now stated

$$\frac{1}{\varrho}\overline{p\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}=-k_p\frac{\sqrt{E}}{L}\left(\overline{u_i u_j}-\frac{2}{3}\delta_{ij}E\right), \tag{3.7}$$

which is valid for i = j as well as for i ≠ j.

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In an inhomogeneous base flow, Eq. (3.7) does not cover all of the effects. Multiplying Eq. (2.18) by $\partial u_i/\partial x_j$ and averaging yields:

$$\begin{aligned}\frac{1}{\varrho}\overline{p\frac{\partial u_i}{\partial x_j}}(\mathfrak{r})=&\frac{1}{2\pi}\sum_{l=1}^{3}\sum_{m=1}^{3}\int_{(\mathrm{Vol})}\frac{\partial U_l(\mathfrak{r}+\mathfrak{x})}{\partial x_m}\overline{\frac{\partial u_m}{\partial x_l}(\mathfrak{r}+\mathfrak{x})\frac{\partial u_i}{\partial x_j}(\mathfrak{r})}\frac{d\,\mathrm{Vol}}{r}+\\&+\frac{1}{4\pi}\sum_{l=1}^{3}\sum_{m=1}^{3}\int_{(\mathrm{Vol})}\overline{\frac{\partial^2 u_l u_m}{\partial x_l\partial x_m}(\mathfrak{r}+\mathfrak{x})\frac{\partial u_i}{\partial x_j}(\mathfrak{r})}\frac{d\,\mathrm{Vol}}{r}.\end{aligned} \tag{3.8}$$

The second integral expression on the right side of this equation approximately represents the contribution covered by the preceding considerations. The first integral expression on the right side of Eq. (3.8) must be emphasized especially. For an estimate of its magnitude, $\partial U_l/\partial x_m$ may be expanded in a Taylor series, as suggested by P. Y. Chou [3]:

$$\begin{aligned}\frac{\partial U_l}{\partial x_m}(\mathfrak{r}+\mathfrak{x})=&\frac{\partial U_l}{\partial x_m}(\mathfrak{r})+\\&+\sum_{s=1}^{\infty}\frac{1}{s!}\sum_{r_1=1}^{3}\sum_{r_2=1}^{3}\cdots\sum_{r_s=1}^{3}\frac{\partial^{s+1}U_l(\mathfrak{r})}{\partial x_{r_1}\partial x_{r_2}\dots\partial x_{r_s}\partial x_m}\xi_{r_1}\xi_{r_2}\dots\xi_{r_s}.\end{aligned} \tag{3.9}$$

The first expression on the right side of Eq. (3.8) can now be integrated by terms, if the $\overline{u_m(\mathfrak{r}+\mathfrak{x})u_i(\mathfrak{r})}$ correlation between the velocity components effective simultaneously at two different locations of space, is known, so that this contribution can be brought into the following form:

$$\begin{aligned}\frac{1}{\varrho}\overline{p\frac{\partial u_i}{\partial x_j}}=&\sum_{l=1}^{3}\sum_{m=1}^{3}\frac{\partial U_l}{\partial x_m}a_{lj}^{mi}+\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\partial^2 U_l}{\partial x_m\partial x_n}{}_n b_{lj}^{mi}+\\&+\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\sum_{k=1}^{3}\frac{\partial^3 U_l}{\partial x_m\partial x_n\partial x_k}{}_{nk}c_{lj}^{mi}+\cdots\end{aligned} \tag{3.10}$$

Estimates indicate that this series expansion as a rule converges well. Under the assumption, rather well satisfied by most flows, that approximately

$$\overline{u_m(\mathfrak{r}+\mathfrak{x})\,u_i(\mathfrak{r})}\approx\overline{u_m(\mathfrak{r})\,u_i(\mathfrak{r}-\mathfrak{x})}$$

the following relationship follows from Green's theorem (see Appendix) for values of a

$$\frac{\partial}{\partial t}(\delta\overline{u_i u_j})=-k_p\frac{\sqrt{E}}{L}\delta\overline{u_i u_j}. \tag{3.11}$$

and the continuity condition yields: /558

$$\sum_{i=1}^{3} a_{li}^{mi}=0; \quad \sum_{m=1}^{3} a_{lm}^{mi}=0; \quad \sum_{i=1}^{3} {}_{n}b_{li}^{mi}=0; \quad \sum_{m=1}^{3} {}_{n}b_{lm}^{mi}=0;$$
$$\sum_{i=1}^{3} {}_{nk}c_{li}^{mi}=0; \quad \sum_{m=1}^{3} {}_{nk}c_{lm}^{mi}=0; \quad \text{etc.} \tag{3.12}$$

The l and j indices are interchangeable in the case of a, b and c; the following is valid for the interchangeability of m and i:

$$a_{lj}^{im}=a_{lj}^{mi}; \quad {}_{nk}c_{lj}^{im}={}_{nk}c_{lj}^{im};$$
but
$${}_{n}b_{lj}^{im}=-{}_{n}b_{lj}^{mi}. \tag{3.13}$$

These relationships may be useful in the estimation of the values of a, b and c, if the $\overline{u_m(\mathfrak{x}+\mathfrak{r})u_i(\mathfrak{x})}$ function is not accurately known. Let us add certain values valid for isotropic turbulence (for derivation see Appendix):

$$a_{ii}^{ii}=0{,}4\,\overline{u_i^2}; \quad a_{jj}^{ii}=0{,}8\,\overline{u_i^2}; \quad {}_{n}b_{lj}^{mi}=0;$$
independent of Reynolds number

$$\begin{aligned} &{}_{ii}c_{ii}^{ii}=-0{,}52\,\overline{u_i^2}\,L^2; \quad {}_{jj}c_{jj}^{ii}=-2{,}28\,\overline{u_i^2}\,L^2;\\ &{}_{jj}c_{ii}^{ii}=-0{,}104\,\overline{u_i^2}\,L^2; \quad {}_{ii}c_{jj}^{ii}=+0{,}62\,\overline{u_i^2}\,L^2;\\ &{}_{kk}c_{jj}^{ii}=+0{,}207\,\overline{u_i^2}\,L^2; \end{aligned}$$
for large Reynolds numbers

$$\begin{aligned} &{}_{ii}c_{ii}^{ii}=-0{,}243\,\overline{u_i^2}\,L^2; \quad {}_{jj}c_{jj}^{ii}=-1{,}07\,\overline{u_i^2}\,L^2;\\ &{}_{jj}c_{ii}^{ii}=-0{,}049\,\overline{u_i^2}\,L^2; \quad {}_{ii}c_{jj}^{ii}=+0{,}292\,\overline{u_i^2}\,L^2;\\ &{}_{kk}c_{jj}^{ii}=+0{,}097\,\overline{u_i^2}\,L^2. \end{aligned}$$
for small Reynolds numbers

As the reference length L for turbulence, the integral over the coefficient

$$g(r)=\frac{\overline{u_i\,u_i'}}{\sqrt{\overline{u_i^2}\;\overline{u_i'^2}}} \tag{3.14}$$

of the correlation between the parallel velocity components at points 0 and 0', at distance r perpendicularly to this velocity component (Figure 4) has been selected:

$$L=\int_0^\infty g(r)\,dr. \tag{3.15}$$

4. The Dissipation of Energy /559

In the expressions

$$2\nu\sum_{k=1}^{3}\overline{\frac{\partial u_i}{\partial x_k}\cdot\frac{\partial u_j}{\partial x_k}} \quad \text{and} \quad \nu\sum_{k=1}^{3}\overline{\left(\frac{\partial u_i}{\partial x_k}\right)^2}$$

in Eqs. (2.10) and (2.11) the effect of viscosity is described; in essence, this consists of the dissipation of energy. Let us first discuss the boundary cases of an infinitely large Reynolds number $Re = \sqrt{E}L/\nu$ and of an infinitely small Re number.

In the case of very large Reynolds numbers the turbulent motion consists of elements of numerous different orders of magnitude. The kinetic energy is contained mainly in the large elements of turbulence, while the most important contributions to $\overline{(\partial u_i/\partial x_k)^2}$ are provided by the small elements. However, all turbulence which is anisotropic in large elements, is isotropic in small elements, in the case of large Re numbers. The causes of this phenomenon, which represents one of the most important results of more recent turbulence research, has been explained, e.g., by C. F. v. Wizsacker [4]. It may also be said, with A. N. Kolmogoroff [5], [6], that the turbulence is locally isotropic, i.e., if the variations of the velocity vector at an arbitrary point, with respect to the instantaneous velocity at a fixed point in space, is viewed, the variations will be isotropic in their statistical average, provided that the points considered are within a sufficiently small range. The cause of this behavior is to be found in the characteristic of turbulence described in Section 3, according to which an isotropic variation distribution is most probable if no external effects are effective. External effects affect almost exclusively large elements only, so that the small elements are isotropic.

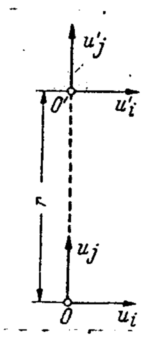

Figure 4. Correlation functions in isotropic turbulence.

$$f(r) = \frac{\overline{u_j u'_j}}{\sqrt{\overline{u_j^2}\,\overline{u_j'^2}}}; \quad g(r) = \frac{\overline{u_i u'_i}}{\sqrt{\overline{u_i^2}\,\overline{u_i'^2}}}$$

Since thus $\sum_{k=1}^{3} \overline{(\partial u_i/\partial x_k)^2}$ is essentially produced by small elements, the term is identical for all three indices i = 1, 2, 3. Total dissipation S in the case of large Re numbers (see e.g. L. Prandtl [1]) then is

$$S = \nu \sum_{i=1}^{3} \sum_{k=1}^{3} \overline{\left(\frac{\partial u_i}{\partial x_k}\right)^2} = c\,\frac{E^{\frac{3}{2}}}{L} \quad [3] \qquad (4.1)$$

3. The terms $\nu \sum_{i=1}^{3} \sum_{k=1}^{3} \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_i}}$ which appear in the general case, disappear due to isotropy.

where c is a dimensionless number which depends to a limited degree on the structure of the turbulence. Then, for large Re /560

$$\nu \sum_{k=1}^{3} \overline{\left(\frac{\partial u_i}{\partial x_k}\right)^2} = \frac{c}{3} \frac{E^{\frac{3}{2}}}{L}. \tag{4.2}$$

Due to local isotropy, in the case of small elements, no correlation can exist between $\partial u_i/\partial x_k$ and $\partial u_j/\partial x_k$. For this reason, the $\sum_{k=1}^{3} \overline{(\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k})}$ expressions are becoming arbitrarily small with respect to $\sum_{k=1}^{3} \overline{(\frac{\partial u_i}{\partial x_k})^2}$. It is therefore permissible to set, with sufficiently large Re numbers, for $i \neq j$:

$$\nu \sum_{k=1}^{3} \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}} = 0 \tag{4.3}$$

Behavior in the case of extremely small Reynolds numbers is fundamentally different. Turbulence here consists of elements which differ little in their magnitude. Elements which contribute the most essential part to energy, also contribute much to $\overline{(\partial u_i/\partial x_k)^2}$ and these elements contributing the most to $\overline{(\partial u_i/\partial x_k)^2}$ proportionally contain a high amount of energy. For this reason, the energy dissipated by the u_i variation component is proportional to the kinetic energy contained in it, i.e., for $Re \to 0$:

$$\sum_{k=1}^{3} \overline{\left(\frac{\partial u_i}{\partial x_k}\right)^2} = \frac{c_1}{2} \frac{\overline{u_i^2}}{L^2}. \tag{4.4}$$

for the same reasons, the $\overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}}$ correlation is also proportional to the value of $\overline{u_i u_j}$, thus

$$\sum_{k=1}^{3} \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}} = \frac{c_1}{2} \frac{\overline{u_i u_j}}{L^2}. \tag{4.5}$$

The magnitude of the c_1 numerical factor in general depends on the choice of the i and j indices, but an approximately uniform order of magnitude can be expected. In the boundary case in which the turbulent motion consists only of sine waves with a single length, one obtains:

$$c_1 = \frac{9\pi^2}{32} = 2.776. \tag{4.6}$$

For isotropic turbulence, in accordance with the theory of G. K. Batchelor and A. A. Townsend [10] or J. Rotta [12] the value of c_1 is:

$$c_1 = \frac{5\pi}{4} = 3.927. \tag{4.7}$$

The mode of transition between these two asymptotic laws can be determined approximately from theoretical and experimental investigations of isotropic turbulence; these also provide some information concerning the magnitude of c in Eqs. (4.1) and (4.2). The value defined by Eq. (3.15) was chosen as the reference length L for the measured results evaluated in Figure 5; the results are of different origin [7] to [11]. The measurements cover a rather broad range of Re numbers. The dependence of dissipation on the Reynolds number can be approximately determined by calculation with the condition that a statistical equilibrium exists for the spectral distribution of the energy of variation [12]. Here, a general constant $\varkappa$ occurs in the determination of turbulent energy exchange processes between elements of different magnitude (see the work of W. Heisenberg [13]). /561

For the purpose of a first approximation, based on Figure 5, the following interpolation formula is proposed:

$$S = \nu c_1 \frac{E}{L^2} + c \frac{E^{\frac{3}{2}}}{L} \tag{4.8}$$

According to Figure 5, approximately c = 0.202 (with Heisenberg's constant $\varkappa$ = 0.28). The transfer of this interpolation formula [Eq. (4.8)] to the expressions of interest would yield

$$\nu \sum_{k=1}^{3} \overline{\left(\frac{\partial u_i}{\partial x_k}\right)^2} = \nu \frac{c_1}{2} \frac{\overline{u_i^2}}{L^2} + \frac{c}{3} \frac{E^{\frac{3}{2}}}{L} \tag{4.9}$$

and generally for i = j and i ≠ j

$$2\nu \sum_{k=1}^{3} \overline{\frac{\partial u_i}{\partial x_k} \cdot \frac{\partial u_j}{\partial x_k}} = \nu c_1 \frac{\overline{u_i u_j}}{L^2} + \delta_{ij} \frac{2c}{3} \frac{E^{\frac{3}{2}}}{L} \tag{4.10}$$

These expressions will be considered valid for later discussions and calculations. /562

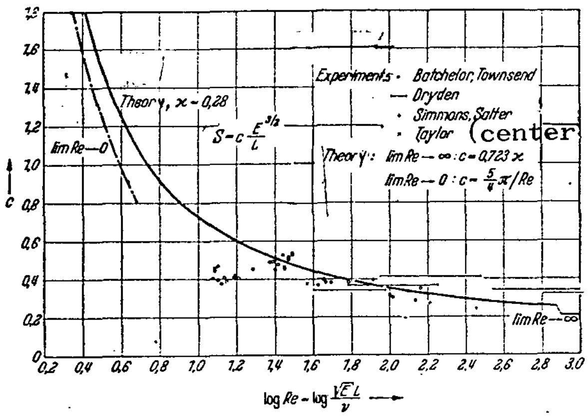

Figure 5. Energy dissipation in isotropic turbulence.

5. Diffusion Terms

If the terms of Eqs. (2.10) and (2.11) are interpreted as $\mathfrak{Q}_{ij}$ vectors with components

$$Q_{ij}^{k} = -\nu\frac{\partial\overline{u_i u_j}}{\partial x_k} + \overline{u_k u_i u_j} + \overline{(\delta_{jk} u_i + \delta_{ik} u_j)\frac{p}{\varrho}} \tag{5.1}$$

then

$$\sum_{k=1}^{3}\frac{\partial}{\partial x_k}\left[-\nu\frac{\partial\overline{u_i u_j}}{\partial x_k} + \overline{u_k u_i u_j} + \overline{(\delta_{jk} u_i + \delta_{ik} u_j)\frac{p}{\varrho}}\right] = \operatorname{div}\mathfrak{Q}_{ij}. \tag{5.2}$$

The terms therefore can be transformed into surface integrals during the integration of Eqs. (2.10), (2.11) and (2.13) over a closed area of space, in accordance with the Gaussian integral theorem. It follows clearly from this characteristic that these terms describe a transport of the statistical characteristic $\overline{u_i u_j}$ and of kinetic energy, briefly, a turbulent diffusion of $\overline{u_i u_j}$, i.e., a diffusion of energy. The processes are caused by a correlation between three velocity components, correlations between pressure and velocity fluctuations and by viscosity effects and have their origin in the spatial inhomogeneity of turbulence. Although in numerous cases the terms are highly important for the overall mechanism of flow processes, there are certain flows or at least certain areas in the flow space in which these diffusions disappear or have only a negligible effect. Let us consider such a flow in more detail in the following section and interrupt the discussion of diffusion terms at this point, to be continued at a later occasion.

6. Turbulent, Parallel Shear Flow

As an example of the application of the relationship given in the foregoing, let us examine a parallel shear flow in which the stationary base flow velocity coincides with the x_1 axis and is a function of x_2 only. To further simplify the problem, the energy diffusion contributions are set equal to 0. Rigorously, of course, this is permissible in special cases only, e.g., in universal turbulent wall flow in which shear stress is constant and viscosity is very low. However, calculations without diffusion terms demonstrate very well the essentials of turbulent shear flow. Measured results obtained by H. Reichardt [14] and J. Laufer [15] in a rectangular channel, make a comparison with theory possible. The case under discussion is realized in the outer areas only, because flow in the central channel zone is affected considerably by the diffusion terms neglected. /563

The energy equations in Eq. (2.11) are reduced to

$$\left.\begin{aligned} \overline{u_1 u_2}\frac{dU_1}{dx_2} - \frac{1}{\varrho}\overline{p\frac{\partial u_1}{\partial x_1}} + \nu\sum_{k=1}^{3}\overline{\left(\frac{\partial u_1}{\partial x_k}\right)^2} &= 0, \\ -\frac{1}{\varrho}\overline{p\frac{\partial u_2}{\partial x_2}} + \nu\sum_{k=1}^{3}\overline{\left(\frac{\partial u_2}{\partial x_k}\right)^2} &= 0, \\ -\frac{1}{\varrho}\overline{p\frac{\partial u_3}{\partial x_3}} + \nu\sum_{k=1}^{3}\overline{\left(\frac{\partial u_3}{\partial x_k}\right)^2} &= 0. \end{aligned}\right\} \quad (6.1)$$

It is seen from the equations that the entire kinetic energy, which is being transferred from the base flow by way of the Reynolds stress $-\overline{\rho u_i u_2}$ into turbulent energy, is transformed directly only into longitudinal fluctuation energy $\rho\overline{u_1^2}/2$. The energy imparted to longitudinal fluctuations is dissipated in part into heat by the u_1 component and in part transferred by the expression $\overline{p\frac{\partial u_i}{\partial x_i}}$ to the u_2 and u_3 components and thus used to maintain the u_2 and u_3 fluctuations, which in their mean are stationary. The fact that a transfer of energy takes place from the u_1 fluctuation to the other fluctuation components, follows from the statement that the $\nu\sum_{k=1}^{3}\overline{(\partial u_1/\partial x_k)^2}$ dissipation contributions can be positive only. The addition of all three equations in Eq. (6.1) yields the balance of the total energy of turbulent motion in accordance with Eq. (2.13)

$$\overline{u_1 u_2}\frac{dU_1}{dx_2} + \nu \sum_{i=1}^{3}\sum_{k=1}^{3}\overline{\left(\frac{\partial u_i}{\partial x_k}\right)^2} = 0. \tag{6.2}$$

In order to determine the correlation function $\overline{u_1 u_2}$, another equation is required; it may be obtained from Eq. (2.10) with i = 1; j = 2:

$$\overline{u_2^2}\frac{dU_1}{dx_2} - \frac{1}{\varrho}\overline{p\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)} + 2\nu \sum_{k=1}^{3}\overline{\frac{\partial u_1}{\partial x_k}\frac{\partial u_2}{\partial x_k}} = 0. \tag{6.3}$$

This equation replaces the conventional expression based on the concept that turbulent shear stress originates in momentum transport proportional to the velocity gradient dU_1/dx_2 /564

$$\overline{u_1 u_2} = -\varepsilon\frac{dU_1}{dx_2}.$$

Here, ε is the so-called exchange factor which has the dimension of kinematic viscosity.

Following the introduction of the statements of Eqs. (3.7), (3.10), (4.8) and (4.10) in Eqs. (6.1), (6.2) and (6.3), the fluctuation squares and the correlation can be calculated, if dU_1/dx_2 and L are considered given values. Specifically, for pressure fluctuations and velocity derivations the following is substituted:

$$\left.\begin{aligned}
\frac{1}{\varrho}\overline{p\frac{\partial u_1}{\partial x_1}} &= a_{11}^{21}\frac{dU_1}{dx_2} + {}_{22}c_{11}^{21}\frac{d^3U_1}{dx_2^3} - \frac{k_p}{2}\frac{\sqrt{E}}{L}\left(\overline{u_1^2} - \frac{2}{3}E\right),\\
\frac{1}{\varrho}\overline{p\frac{\partial u_2}{\partial x_2}} &= a_{12}^{22}\frac{dU_1}{dx_2} + {}_{22}c_{12}^{22}\frac{d^3U_1}{dx_2^3} - \frac{k_p}{2}\frac{\sqrt{E}}{L}\left(\overline{u_2^2} - \frac{2}{3}E\right),\\
\frac{1}{\varrho}\overline{p\frac{\partial u_3}{\partial x_2}} &= a_{13}^{23}\frac{dU_1}{dx_2} + {}_{22}c_{13}^{23}\frac{d^3U_1}{dx_2^3} - \frac{k_p}{2}\frac{\sqrt{E}}{L}\left(\overline{u_3^2} - \frac{2}{3}E\right),\\
\frac{1}{\varrho}\overline{p\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)} &= (a_{12}^{21} + a_{11}^{22})\cdot\frac{dU_1}{dx_2} +\\
&\quad + ({}_{22}c_{12}^{21} + {}_{22}c_{11}^{22})\frac{d^3U_1}{dx_2^3} - k_p\frac{\sqrt{E}}{L}\overline{u_1 u_2}.
\end{aligned}\right\} \tag{6.4}$$

The following considerations are used to estimate the values of a_{1j}^{2i} and ${}_{22}c_{1j}^{2i}$: Due to Eq. (3.11), initially $\sum_{i=1}^{3} a_{ii}^{21} = 2\overline{u_1 u_2}$ is valid; because according to experience the u_1 fluctuation is larger on the average than the u_2 fluctuation, the following estimate is made, based on results obtained for isotropic turbulence:

$$a_{11}^{21} = 0{,}4\,\overline{u_1 u_2} \tag{6.4a}$$

and accordingly, for large Reynolds numbers

$$_{22}c_{11}^{21} = -0.104\,\overline{u_1 u_2}\,L^2. \qquad (6.4b)$$

It follows further from the continuity condition in Eq. (3.12)

$$a_{12}^{22} + a_{13}^{23} = -0.4\,\overline{u_1 u_2},$$
$$_{22}c_{12}^{22} + {}_{22}c_{13}^{23} = +0.104\,\overline{u_1 u_2}\,L^2.$$

Since in the present case no additional information is available concerning the structure of turbulence, let us set:

$$\left.\begin{aligned} a_{12}^{22} &= a_{13}^{23} = -0.2\,\overline{u_1 u_2}, \\ _{22}c_{12}^{22} &= {}_{22}c_{13}^{23} = 0.052\,\overline{u_1 u_2}\,L^2. \end{aligned}\right\} \qquad (6.4c)$$

This simplification renders the equations of the fluctuation energy of the u_2 and u_3 components identical in accordance with Eq. (6.1), so that $\overline{u_2^2} = \overline{u_3^2}$. /565

Calculations for isotropic turbulence yield:

$$a_{11}^{22} = 0.8\,\overline{u_2^2}; \qquad _{22}c_{11}^{22} = 0.62\,\overline{u_2^2}\,L^2. \qquad (6.4d)$$

To estimate a_{12}^{21}, the continuity expressed in Eq. (3.12) and the exchange rules of Eq. (3.13) must be used:

$$a_{11}^{11} + a_{12}^{12} + a_{13}^{13} = 0,$$
$$a_{21}^{21} + a_{22}^{22} + a_{23}^{23} = 0,$$
$$a_{31}^{31} + a_{32}^{32} + a_{33}^{33} = 0.$$

Since, due to the simplification made in Eq. (6.4c) leading to $\overline{u_2^2} = \overline{u_3^2}$, $a_{22}^{22} = a_{33}^{33}$ will also be true, it follows from the same system of equation that

$$a_{12}^{21} = -\tfrac{1}{2}\,a_{11}^{11} = -0.2\,\overline{u_1^2}. \qquad (6.4e)$$

The same calculations finally yield:

$$_{22}c_{12}^{21} = -\tfrac{1}{2}\,{}_{22}c_{11}^{11} = 0.052\,\overline{u_1^2}\,L^2. \qquad (6.4f)$$

Prior to attempting solutions of the Eq. (6.1) system, let us examine the extent to which available measurements qualitatively confirm the theory and its simplifications. The experiments of [14], [15] indicate substantially greater values for the u_1 fluctuations than for the rest of the components. Since it was concluded rigorously from Eq. (6.1) that a transfer of energy takes place from u_1 to u_2 and u_3, an exchange of energy occurs, in accordance with statements in Section 3, from higher intensity fluctuation components to those of lower intensity. In addition, the equality of the fluctuation squares $\overline{u_2^2}$ and $\overline{u_3^2}$ has been confirmed by the experiments of J. Laufer [15]. Finally, the spectra measured by J. Laufer confirm the existence of local isotropy. These spectra exhibit a certain fluctuation intensity above a certain frequency, but no correlation which indicates isotropy in this frequency range. The relationship $\overline{(\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_1})/(\frac{\partial u_1}{dx_1})^2} \sim \frac{1}{8}$ calculated from the second momentum of the spectra, however, suggests that the magnitude of $v \sum_{k=1}^{3} \overline{\frac{\partial u_1}{\partial x_k}\frac{\partial u_2}{\partial x_k}}$ is in fact greater than the amount obtained by calculation from Eq. (4.10) with corresponding Reynolds numbers. In addition, local isotropy was shown experimentally to exist in the turbulent wake of a cylinder by A. A. Townsend [16] and at the edge of a cylindrical jet by S. Corrsion [17]. It also follows from the theory of local isotropy that with large Re numbers one-third of the energy introduced by the longitudinal fluctuation u_1 is dissipated immediately, while two-thirds are transferred by the term $p\frac{\partial u_1}{\partial x_1}$ to the other components. /566

Let us next apply the calculations to the case of very large Reynolds numbers where terms depending on viscosity can be eliminated. Let us investigate the universal wall flow in which the well-known logarithmic law is valid between the base flow velocity U_1 and the wall distance x_2 so that the following relationship exists between the third and first derivation

$$\frac{d^3U_1}{dx_2^3} = \frac{2}{x_2^2}\frac{dU_1}{dx_2}$$

The second equation in Eq. (6.1) -- after elimination of $\overline{u_1u_2}\frac{dU_1}{dx_2}$ -- then yields with Eq. (6.2)

$$\frac{\overline{u_2^2}}{E} = \frac{2}{3}\left[1 - \frac{c}{k_p}\left(0{,}4 + 0{,}312\frac{L^2}{x_2^2}\right)\right]. \tag{6.5}$$

This provides the first indication of the magnitude of k_p, of which nothing has been known so far. Since $\overline{u_2^2}$ can only be positive, $k_p > 0.4\ c$ must be true. For $\overline{u_1^2}$ then

$$\frac{\overline{u_1^2}}{E} = \frac{2}{3}\left[1 + 2\frac{c}{k_p}\left(0{,}4 + 0{,}312\frac{L^2}{x_2^2}\right)\right]. \tag{6.6}$$

Substitution of Eqs. (6.5) and (6.6) in Eqs. (6.3) and (6.2) yields, after some calculation

$$\frac{\tau}{\varrho} = -\overline{u_1u_2} = \left(\frac{2}{3k_p}\right)^{\frac{3}{2}}\left\{\left(0{,}2 - 1{,}24\frac{L^2}{x_2}\right)\left[1 - \frac{c}{k_p}\left(0{,}4 + 0{,}312\frac{L^2}{x_2^2}\right)\right] + \right.$$
$$\left. + \left(0{,}2 - 0{,}1\frac{L^2}{x_2^2}\right)\left[1 + 2\frac{c}{k_p}\left(0{,}4 + 0{,}312\frac{L^2}{x_2^2}\right)\right]\right\}^{\frac{3}{2}}\frac{1}{\sqrt{c}}L^2\left|\frac{dU_1}{dx_2}\right|\frac{dU_1}{dx_2}. \tag{6.7}$$

For L/x_2 = constant, this formula conforms entirely with the Prandtl mixing path theorem. If the length L is set so that it equals the mixing path, i.e. $L \sim 0.4x_2$, then:

thus

$$\left.\begin{aligned} \left(0{,}122\,\frac{1+0.9\,\frac{c}{k_p}}{k_p/c}\right)^{\frac{3}{2}}\frac{1}{c^2} &= 1, \\ c &= \left[0{,}122\,\frac{c}{k_p}\left(1+0{,}9\,\frac{c}{k_p}\right)\right]^{\frac{3}{4}}. \end{aligned}\right\} \qquad (6.8)$$

This definition of L thus establishes a relationship between the values of c and k_p. If a certain value of c/k_p is assumed ($c/k_p < 2.5$), then c and thus k_p may be obtained from Eq. (6.8). Such calculated results are compiled in Table 1 and thus important comparative values are given for use with experimental values. /567

TABLE 1

c/k_p	c	k_p	$\frac{\sqrt{\overline{u_1^2}}}{\sqrt{\tau/\varrho}}$	$\sqrt{\frac{\overline{u_2^2}}{\overline{u_1^2}}} = \sqrt{\frac{\overline{u_3^2}}{\overline{u_1^2}}}$	$-\frac{\overline{u_1 u_2}}{\sqrt{\overline{u_1^2}\,\overline{u_2^2}}}$
0,5	0,162	0,324	1,802	0,731	0,421
0,6	0,194	0,323	1,750	0,689	0,475
0,7	0,228	0,326	1,708	0,648	0,530
0,8	0,262	0,327	1,674	0,610	0,585
0,9	0,298	0,321	1,646	0,573	0,644

An experimental determination of the cross correlation function in a tube by G. I. Taylor [11] yielded for the tube center $L = \int_0^\infty g(r)\,dr$ = 0.14 tube radius; this value corresponds exactly to the magnitude of the mixing length as evaluated by J. Nikuradse [18] so that it may be assumed that the value of L in accordance with Eq. (3.15) and the mixing path l at high Reynolds numbers are actually of the same order of magnitude. According to Table 1, the c factor is generally of the order of magnitude of the evaluation results of isotropic turbulence (Figure 5). The evaluations performed by K. Wiechardt in connection with the work of L. Prandtl [1], yielded $c \sim 0.18$ to 0.21. Boundary layer measurements showed that the value of $\sqrt{\overline{u_1^2}}/\sqrt{\tau/\varrho}$ is about 1.7 to 1.18 (see also J. Rotta [19]); the measurements of [14], [15] did not yield uniform results in this respect, peak values are in part around $\sqrt{\overline{u_1^2}}/\sqrt{\tau/\varrho} > 2$. The ratio of the fluctuation components $\sqrt{\overline{u_2^2}}/\sqrt{\overline{u_1^2}}$, according to H. Reichardt [14] and J.

Laufer [15] is 1/2 to 1/3. According to the measurements cited, [14], [15], the correlation coefficient is 0.45 to 0.5. Taking all of the experimental results into consideration, calculations with $c/k_p = 0.7$ appears to agree best with actual conditions. In any case, the statement may be made that the calculated results are overall in approximate agreement with experimental observations, both qualitatively and quantitatively.

With inclusion of the viscosity and a Reynolds number of $Re = \sqrt{E}L/v$ of turbulence, the following expressions are obtained in the case of a simple parallel shear flow (d^2U_1/dx_2^2 and higher derivatives are set equal to 0), for the most important values: /568

$$-\overline{u_1u_2} = \frac{\left[\frac{0.4}{3}\,\frac{2+0.4\frac{c}{k_p}+2.4\frac{c_1/k_p}{Re}}{k_p}\right]^{\frac{3}{2}}}{\left(1+\frac{c_1/k_p}{Re}\right)^3\sqrt{c+\frac{c_1}{Re}}}\,L^2\left|\frac{dU_1}{dx_2}\right|\frac{dU_1}{dx_2}, \tag{6.9}$$

$$Re+\frac{c_1}{k_p} = \sqrt{\frac{0.4}{3}\,\frac{2+0.4\frac{c}{k_p}+2.4\frac{c_1/k_p}{Re}}{k_p\left(c+\frac{c_1}{Re}\right)}\,\frac{L^2\frac{dU_1}{dx_2}}{v}}. \tag{6.10}$$

From these equations $\overline{-u_1u_2}$ can be determined, if dU_1/dx_2, L and v are given. The execution of the calculation was based on the value of $c_1 = 5\pi/4$, obtained theoretically for isotropic turbulence; it was confirmed experimentally in wind tunnel experiments [10]. It was assumed, on the other hand, that k_p is independent of the Reynolds number, although a certain dependence is generally conceivable. It may be assumed that k_p retains at least the same order of magnitude over the entire Reynolds number range.

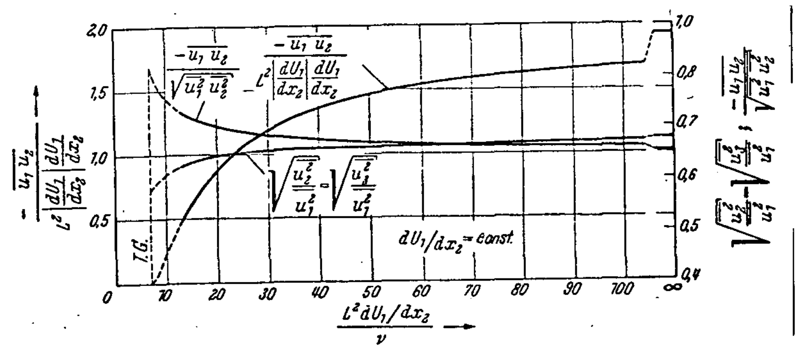

Figure 6. Statistical values and correlation of velocity fluctuations in parallel shear flow.

Figure 6 summarizes the results of such calculations in dimensionless form, by plotting $-\overline{u_1 u_2} / L^2 \left|\frac{dU_1}{dx_2}\right| \frac{dU_1}{dx_2}$ over the Re number $\frac{L^2}{\nu} \frac{dU_1}{dx_2}$. In the diagram the case of Re $\rightarrow \infty$ is represented by the straight line $-\overline{u_1 u_2} / L^2 \left|\frac{dU_1}{dx_2}\right| \frac{dU_1}{dx_2} =$ 1.93. With decreasing Reynolds numbers $-\overline{u_1 u_2}$ declines to 0. The remarkable result that below a certain Reynolds number $\frac{L^2 \frac{dU_1}{dx_2}}{\nu}$ no stationary turbulent state of equilibrium can exist, can also be derived from the total energy equation in Eq. (6.2). According to the Cauchy-Schwarz inequality /569

$$(\overline{u_1 u_2})^2 < \overline{u_1^2}\,\overline{u_2^2}.$$

Since the dissipation in Eq. (6.2) is always positive, $\overline{u_1 u_2}$ is always negative with a positive dU_1/dx_2. Further, because of Eq. (2.12)

$$E > \frac{\overline{u_1^2} + \overline{u_2^2}}{2}; \qquad \text{i.e.} \quad \overline{u_2^2} < 2E - \overline{u_1^2}$$

thus

$$(\overline{u_1 u_2})^2 < \overline{u_1^2}(2E - \overline{u_1^2}) < E^2.$$

Therefore, if dU_1/dx_2 is positive:

$$\sum_{i=1}^{3} \sum_{k=1}^{3} \overline{\left(\frac{\partial u_i}{\partial x_k}\right)^2} > c_1 \frac{E}{L^2},$$

According to Eq. (4.8)

it then follows from Eq. (6.2) that

$$E \frac{dU_1}{dx_2} > -\overline{u_1 u_2} \frac{dU_1}{dx_2} = \nu \sum_{i=1}^{3} \sum_{k=1}^{3} \overline{\left(\frac{\partial u_i}{\partial x_k}\right)^2} > \nu c_1 \frac{E}{L^2}, \tag{6.11}$$

so that as a rough estimate of the boundary of a possible turbulent flow state which is stationary in its mean, the following is obtained

$$\left(\frac{L^2 \frac{dU_1}{dx_2}}{\nu}\right)_{TB} > c_1 \tag{6.12}$$

The value of this estimate lies in the fact that most of the assumptions of Sections 3 and 4 can be eliminated in the confirmation of the qualitative correctness of the curves shown in Figure 6. A more accurate value for the "turbulence boundary" is obtained from Eq. (6.10) for lim Re $\rightarrow$ 0 as

$$\left(\frac{L^2 \frac{dU_1}{dx_2}}{\nu}\right)_{TB} = 1.77\, c_1 = 6.94. \tag{6.13}$$

Below this turbulence limit only laminar flow is possible, while above the limit both laminar and turbulent flow may exist. Under certain conditions, this fact is of importance with respect to the problem of the transition from laminar to turbulent flow.

In Figure 6 the correlation coefficient /570

$$-\frac{\overline{u_1 u_2}}{\sqrt{\overline{u_1^2}\,\overline{u_2^2}}} = \frac{3}{2}\sqrt{\frac{\frac{0{,}4}{3}\left(2+0{,}4\frac{c}{k_p}+2{,}4\frac{c_1/k_p}{Re}\right)\left(\frac{c}{k_p}+\frac{c_1/k_p}{Re}\right)}{\left(1+0.8\frac{c}{k_p}+1.8\frac{c_1/k_p}{Re}\right)\left(1-0{,}4\frac{c}{k_p}+0{,}6\frac{c_1/k_p}{Re}\right)}} \tag{6.14}$$

and the relationship

$$\sqrt{\frac{\overline{u_2^2}}{\overline{u_1^2}}} = \sqrt{\frac{1-0{,}4\frac{c}{k_p}+0{,}6\frac{c_1/k_p}{Re}}{1+0{,}8\frac{c}{k_p}+1{,}8\frac{c_1/k_p}{Re}}} \tag{6.15}$$

are also plotted over the Reynolds number $\frac{L^2}{v}\frac{dU_1}{dx_2}$. The correlation coefficient increases with decreasing Reynolds numbers, which has been confirmed qualitatively by the measurements of J. Laufer [15]; $\sqrt{\overline{u_2^2}\,/\,\overline{u_1^2}}$ on the other hand, declines with the Reynolds number.

Appendix

Let us designate the correlation between the fluctuations of the velocity component u_i at point $O(\mathfrak{x})$ and u_m at $O'(\mathfrak{x}+\mathfrak{r})$

$$R_i^m(\mathfrak{x},\mathfrak{r}) = \overline{u_i(\mathfrak{x})\,u_m(\mathfrak{x}+\mathfrak{r})} \tag{A.1}$$

In homogeneous turbulence, in which all statistical characteristics are independent of the position vector $\mathfrak{x}$, this function is dependent on the vector $\mathfrak{r}$ only and differentiation yields the well-known expression (see Karman & Howarth [20]):

$$\overline{\frac{\partial u_i(\mathfrak{x})}{\partial x_j}\,\frac{\partial u_m(\mathfrak{x}+\mathfrak{r})}{\partial x_l}} = -\frac{\partial^2 R_i^m(\mathfrak{r})}{\partial \xi_j\,\partial \xi_l}. \tag{A.2}$$

where ξ_i are the components of $\mathfrak{x}$. This relationship may be considered as a rule approximately valid in the general case of inhomogeneous turbulence also. The coefficients of Eq. (3.10) thus are

$$a_{lj}^{m\,i} = -\frac{1}{2\pi}\int\limits_{(\mathrm{Vol})}\frac{\partial^2 R_i^m(\mathfrak{r})}{\partial\xi_l\,\partial\xi_j}\,\frac{d\,\mathrm{Vol}}{r}, \tag{A.3}$$

$$
{}_n b^{mi}_{lj} = -\frac{1}{2\pi}\int\limits_{(\mathrm{Vol})} \frac{\partial^2 R_i^m(\mathfrak{r})}{\partial\xi_l\,\partial\xi_j}\,\xi_n\,\frac{d\,\mathrm{Vol}}{r}, \qquad \text{(A.4)}
$$

$$
{}_{nk} c^{mi}_{lj} = -\frac{1}{4\pi}\int\limits_{(\mathrm{Vol})} \frac{\partial^2 R_i^m(\mathfrak{r})}{\partial\xi_l\,\partial\xi_i}\,\xi_n\,\xi_k\,\frac{d\,\mathrm{Vol}}{r}. \qquad \text{(A.5)}
$$

(A.3) further yields /571

$$
\sum_{j=1}^{3} a^{mi}_{jj} = -\frac{1}{2\pi}\int\limits_{(\mathrm{Vol})}\left(\frac{\partial^2}{\partial\xi_1^2}+\frac{\partial^2}{\partial\xi_2^2}+\frac{\partial^2}{\partial\xi_3^2}\right)R_i^m(\mathfrak{r})\,\frac{d\,\mathrm{Vol}}{r}. \qquad \text{(A.6)}
$$

According to Green's theorem, it follows from Eq. (3.11) that

$$
R_i^m(\mathfrak{r}) = A(r)\,\frac{\xi_i\xi_m}{r^2} + \delta_{im}B(r), \qquad \text{(A.7)}
$$

The ${}_n b^{mi}_{lj}$ can have values other than 0 only if the $R_i^m(\mathfrak{r})$ function is not symmetrical. With the relationship

(A.8)

which is rigorously valid for homogeneous turbulence, but only approximately for inhomogeneous turbulence, Eqs. (A.3), (A.4) and (A.5) directly yield the exchange rules for the indices given in Eq. (3.13).

In the case of isotropic turbulence, the correlation function [20] is:

(A.9)

where (Figure 4)

$$
\overline{u_i^2}\,f(r) = \int_0^\infty \frac{2F(k)}{(rk)^2}\left(\frac{\sin rk}{rk} - \cos rk\right)dk. \qquad \text{(A.10)}
$$

The f(r) function is related to the energy spectrum F(k) in the following manner [13]; here k is the wave number

$$
\left.\begin{aligned}
A(r) &= \overline{u_i^2}\,[f(r)-g(r)] = -\frac{\overline{u_i^2}}{2}\,r\,\frac{df}{dr}\\
B(r) &= \overline{u_i^2}\,g(r) = \overline{u_i^2}\,f(r) - A(r)
\end{aligned}\right\} \qquad \text{(A.11)}
$$

The different integral moments which occur in the squares of Eqs. (A.3) and (A.5), are calculated by

$$
\int_0^\infty A(r)\,\frac{dr}{r} = -\frac{\overline{u_i^2}}{2}\int_0^\infty \frac{df(r)}{dr}\,dr = \frac{\overline{u_i^2}}{2}\,f(0) = \frac{\overline{u_i^2}}{2}, \qquad \text{(A.12)}
$$

$$\int_0^\infty A(r)\,r\,dr = -\frac{\overline{u_i^2}}{2}\int_0^\infty r^2\frac{df(r)}{dr}\,dr = \overline{u_i^2}\int_0^\infty f(r)\,r\,dr \qquad \text{(A.13a)}$$

and further with Eq. (A.11)

$$\int_0^\infty A(r)\,r\,dr = 2\int_0^\infty \frac{F(k)}{k^2}\,dk, \qquad \text{(A.13b)}$$

$$\int_0^\infty B(r)\,r\,dr = \overline{u_i^2}\int_0^\infty f(r)\,r\,dr - \int_0^\infty A(r)\,r\,dr = 0. \qquad \text{(A.14)}$$

The value of Eq. (A.12) is independent of the special form of the correlation /572
function f(r) and thus of the Re number. The values of a_{1j}^{mi} therefore do not depend on the Reynolds number in the case of isotropic turbulence. The Eq. (A.13) expression, on the other hand, depends on the energy spectrum and thus on the Reynolds number Re = $\frac{\sqrt{E}L}{\nu}$. For very large Reynolds numbers, where the spectrum follows the form $\lim_{k\to 0} F(k) \propto k^4$ for small wave numbers and the form $F(k) \propto k^{-5/3}$ for large wave numbers, based on the calculations of J. Rotta [12], the following is valid

$$\int_0^\infty \frac{F(k)}{k^2}\,dk = 2{,}72\,\overline{u_i^2}L^2. \qquad \text{(A.15)}$$

For small Re numbers, where the spectrum has the form of

$$F(k) = \frac{128}{\pi^3}\,\overline{u_i^2}L^5k^4e^{-4/\pi L^2k^2}$$

$$\int_0^\infty \frac{F(k)}{k^2}\,dk = \frac{4}{\pi}\,\overline{u_i^2}L^2. \qquad \text{(A.16)}$$

If Eqs. (A.3), (A.4) and (A.5) are based on the statement of Eqs. (A.9) and (A.10), then the squares, with the use of Eqs. (A.12), (A.13), (A.14), (A.15) and (A.16) lead to the values given in Section 3.

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